Some Properties of the Poisson Distribution

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This paper lists various properties of the Poisson distribution and related functions which can be derived from elementary principles without reference to theories of probability or statistics. They are intended for use by persons who may have to deal with Poisson distributed variables but not from a satistical point of view. Limit values and sums of the Poisson distribution, the sums of related functions, and different relations between its sums and integrals are given.

Differential and difference equations which lead to solutions in the form of a Poisson distribution are discussed. The general elution equation is derived by setting up the differential-difference equation for plate n in a chromatographic column and showing that the Poisson distribution is a solution to this equation. The complete solution is then obtained by applying the boundary conditions of the process.

The function $e^{-u}(u^n/n!)$ where n is a positive integer and u is a positive quantity is called the *Poisson distribution*, and the function $\sum_{r=n}^{\infty} e^{-u}(u^r/r!)$ is called the *Poisson summation distribution*. In a previous paper by this author (*1*) it was shown that if a chromatographic column is regarded as equivalent to a finite number of equilibrium stages or theoretical plates, the distribution of a zone on these plates during its elution would be in the form of a Poisson or a Poisson summation distribution depending upon whether the initial zone occupied one or more than one plate.

Owing to its applications in chromatography and because it has many interesting properties and promises to be of help in solving other problems in chemical engineering unit operations, the Poisson distribution was further investigated, and in this paper some of its properties are summarized. It is also applied to the solution of the differential-difference equation which represents the solute-concentration distribution on a chromatographic column during elution. Such equations of the combined type may become useful in solving other chemical engineering problems, and this author feels that development of the mathematics needed for the solution of these equations may prove valuable to the chemical engineer.

If the function e^u is expanded in a Machlaurin series one gets

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots$$

multiplying both sides by e^{-u}

$$\therefore\ 1 = e^{-u} + e^{-u}u + e^{-u}\frac{u^2}{2!} + e^{-u}\frac{u^3}{3!} + \cdots$$

$$= \phi_0^u + \phi_1^u + \phi_2^u + \cdots$$

$$= \sum_{r=0}^{\infty} \phi_0^u = P_0^u$$

SOME LIMIT VALUES OF ϕ_n^u AND P_n^u

$$\phi_n^0 = 0 \text{ for } n > 0 \qquad (1)$$

$$\lim_{u\to 0} \phi_0^u = 1 \qquad (2)$$

$$\phi_0^u = e^{-u} \qquad (3)$$

$$\phi_n^\infty = 0 \qquad (4)$$

$$\phi_\infty^u = 0 \qquad (5)$$

$$P_0^u = 1 \qquad (6)$$

$$P_n^0 = 0 \text{ for } n > 0 \qquad (7)$$

$$\lim_{u\to 0} P_0^u = 1 \qquad (8)$$

$$P_n^\infty = 1 \qquad (9)$$

$$P_\infty^u = 0 \qquad (10)$$

$$\lim_{n\to\infty} n^{(m)}\phi_n^u = 0 \qquad (11)$$

$$\lim_{u\to\infty} u^m\phi_n^u = 0 \qquad (12)$$

$$\lim_{n\to\infty} n^{(m)}P_n^u = 0 \qquad (13)$$

where

$$n^{(m)} = \frac{n!}{(n-m)!} = n(n-1)(n-2)\cdots(n-m+1)$$

Some of these relations can be proved directly, but others require the application of theories of limits. In Appendix 1* seven of these relations are deduced by means of theories of limits; the other six are simple and can be deduced directly by elementary principles.

FURTHER PROPERTIES OF THE POISSON DISTRIBUTION

The following properties can be proved by expanding and rearranging; detailed derivations are given in Appendix 2*

$$\sum_{n=1}^{\infty} P_n^u = u \qquad (14)$$

$$\sum_{r=1}^{n} P_r^u = u(1 - P_n^u) + nP_{n+1}^u \qquad (15)$$

$$\frac{d}{du}P_n^u = \phi_{n-1}^u \qquad (16)$$

or

$$\int \phi_n^u\, du = P_{n+1}^u + C \qquad (16a)$$

$$\sum_{r=0}^{n} \phi_r^{u_1}\phi_{n-r}^{u_2} = \phi_n^{u_1+u_2} \qquad (17)$$

$$\sum_{r=1}^{n} P_r^{u_1}\phi_{n-r}^{u_2} = P_n^{u_1+u_2} - P_n^{u_2} \qquad (18)$$

$$\sum_{r=1}^{n-1} P_r^{u_1}P_{n-r}^{u_2} = (n-1)(1 + P_n^{u_1} + P_n^{u_2} - P_n^{u_1+u_2}) - u_1P_{n-1}^{u_1} - u_2P_{n-1}^{u_2} + (u_1+u_2)P_{n-1}^{u_1+u_2} \qquad (19)$$

when $u_1 = u_2 = u$ one gets

$$\sum_{r=0}^{n} \phi_r^u\phi_{n-r}^u = \phi_n^{2u} \qquad (17a)$$

$$\sum_{r=1}^{n} P_r^u\phi_{n-r}^u = P_n^{2u} - P_n^u \qquad (18a)$$

$$\sum_{r=1}^{n-1} P_r^uP_{n-r}^u = (n-1)(1 + 2P_n^u - P_n^{2u}) - 2u(1 - P_{n-1}^{2u}) \qquad (19a)$$

RELATION BETWEEN THE INTEGRAL AND SUMMATION OF THE POISSON FUNCTION

Symbols and Definitions

For simplicity, the following symbols will be used:

1. $\int^{[1]} P_n^u \equiv \int_0^u P_n^u\, du$

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*See footnote in column 2.

and

$$\int^{[2]} P_n^u \equiv \int_0^u \left(\int^{[1]} P_n^u\right) du$$

and generally

$$\int^{[m]} P_n^u \equiv \int_0^u \left(\int^{[m-1]} P_n^u\right) du$$

$$\equiv \int_0^u \cdots \int_0^u P_n^u (du)^m$$

2. $$\sum^{[1]} P_n^u \equiv \sum_{r=n+1}^{\infty} P_r^u$$

and

$$\sum^{[2]} P_n^u \equiv \sum_{r=n+1}^{\infty} \left(\sum^{[1]} P_r^u\right)$$

and generally

$$\sum^{[m]} P_n^u \equiv \sum_{r=n+1}^{\infty} \left(\sum^{[m-1]} P_r^u\right)$$

$$\equiv \sum_{r_m=n+1}^{\infty} \cdots \sum_{r_2=r_3+1}^{\infty} \sum_{r_1=r_2+1}^{\infty} P_{r_1}^u$$

By means of integration and summation by parts, the following relations between the integral and sum of the Poisson function have been deduced. Details of the derivations are given in Appendix 3*.

a. $$\int_0^u P_n^u\, du = \sum_{r=n+1}^{\infty} P_r^u$$

$$= uP_n^u - nP_{n+1}^u \qquad (20)$$

or according to this notation

$$\int^{[1]} P_n^u = \sum^{[1]} P_n^u$$

$$= uP_n^u - nP_{n+1}^u \qquad (20)$$

and for $n = 0$

$$\int^{[1]} P_0^u = \sum^{[1]} P_0^u = u \qquad (20a)$$

b. $$\int_0^u uP_n^u\, du = \sum_{r=n+1}^{\infty} rP_{r+1}^u$$

$$= \frac{u^2}{2} P_n^u - \frac{n(n+1)}{2} P_{n+2}^u \qquad (21)$$

and for $n = 0$

$$\int_0^u uP_0^u\, du = \sum_{r=1}^{\infty} rP_{r+1}^u = \frac{u^2}{2} \qquad (21a)$$

c. $$\int_0^u u^m P_n^u\, du$$

$$= \sum_{r=n+1}^{\infty} (r + m - 1)^{(m)} P_{r+m}^u$$

$$= \frac{u^{m+1}}{m+1} P_m^u - \frac{(n+m)^{(m+1)}}{m+1} P_{n+m+1}^u \qquad (22)$$

and for $n = 0$

$$\int_0^u u^m P_0^u\, du = \sum_{r=1}^{\infty} (r + m - 1)^{(m)} \cdot P_{r+m}^u = \frac{u^{m+1}}{m+1} \qquad (22a)$$

d. $$\int^{[m]} P_n^u = \sum^{[m]} P_n^u = \frac{1}{m!} \cdot \sum_{r=0}^{m} (-1)^r C_r^m u^{m-r} (n+r-1)^{(r)} P_{n+r}^u \qquad (23)$$

and for $n = 0$

$$\int^{[m]} P_0^u = \sum^{[m]} P_0^u = \frac{u^m}{m!} \qquad (23a)$$

One also deduces the following:

1. when $m = 0$,

$$\int^{[0]} P_n^u = \sum^{[0]} P_n^u = P_n^u \qquad (24)$$

2. $$\frac{d}{du} \int^{[m]} P_n^u = \int^{[m-1]} P_n^u \qquad (25)$$

DIFFERENTIAL AND DIFFERENCE EQUATIONS LEADING TO SOLUTION IN THE FORM OF A POISSON DISTRIBUTION

The functions ϕ_n^u and P_n^u are functions of both u and n. They are continuous functions with respect to u and step functions with respect to n. In other words they are defined for any positive value of u whether integer or not, but they are defined only for values of n which are integer and positive.

If the symbol y is used to denote such a function, then $\partial y/\partial u$ represents the differential of y with respect to u, and n is kept constant; $\Delta_{(n)} y$ or simply Δy represents the first forward difference of the function y with respect to n with u held constant, and $\nabla_{(n)} y$ or ∇y represents the first backward difference of y

$$\Delta y = y_{n+1} - y_n$$

$$\nabla y = y_n - y_{n-1}$$

Also $\Delta_{(n)}{}^r y$ or $\Delta^r y$ represents the rth forward difference of the function y, and $\nabla_{(n)}{}^r y$ or $\nabla^r y$ represents the rth backward difference.

The subscript (n) is omitted when y is a function of n only, or, as in the case of the Poisson function, it is a step function of n only and a continuous function of the other variables. It is evident that the difference is always taken with respect to n

$$\Delta^r y = y_{n+r} - r y_{n+r-1} + \frac{r(r-1)}{2!} y_{n+r-2} - \cdots + (-1)^r y_n$$

$$= \sum_{S=0}^{r} (-1)^S C_S^{r-S+1} y_{n+r-S} \qquad (26)$$

and

$$\nabla^r y = y_n - r y_{n-1} + \frac{r(r-1)}{2!} y_{n-2} - \cdots + (-1)^r y_{n-r}$$

$$= \sum_{S=0}^{r} (-1)^S C_S^{r-S+1} y_{n-S} \qquad (26a)$$

Differentiating both the Poisson and the Poisson summation functions with respect to u, one gets

$$\frac{\partial}{\partial u} \phi_n^u = \phi_{n-1}^u - \phi_n^u = -\nabla \phi_n^u \qquad (27)$$

and

$$\frac{\partial}{\partial u} P_n^u = \phi_{n-1}^u$$

$$= P_{n-1}^u - P_n^u = -\nabla P_n^u \qquad (27a)$$

Differentiating again with respest to u, one finds that

$$\frac{\partial^2}{\partial u^2} \phi_n^u = \phi_{n-2}^u - 2\phi_{n-1}^u + \phi_n^u = \nabla^2 \phi_n^u \qquad (28)$$

and

$$\frac{\partial^2}{\partial u^2} P_n^u = \phi_{n-2}^u - \phi_{n-1}^u = P_{n-2}^u - 2P_{n-1}^u + P_n^u = \nabla^2 P_n^u \qquad (28a)$$

Generally by differentiating r times one can show that

$$\frac{\partial^r}{\partial u^r} \phi_n^u = (-1)^r \nabla^r \phi_n^u \qquad (29)$$

and

$$\frac{\partial^r}{\partial u^r} P_n^u = (-1)^r \nabla^r P_n^u \qquad (29a)$$

When y is a function of u only, then

$$\frac{d\phi_n^u}{du} = \phi_{n-1}^u - \phi_n^u = \phi_n^u \left(\frac{n}{u} - 1\right) \qquad (30)$$

Similarly

$$\frac{dP_n^u}{du} = \phi_{n-1}^u$$

$$\frac{d^2 P_n^u}{du^2} = \phi_{n-2}^u - \phi_{n-1}^u$$

$$= \phi_{n-1}^u \left(\frac{n-1}{u} - 1\right)$$

$$\therefore \frac{d^2 P_n^u}{du^2} + \frac{dP_n^u}{du}\left(1 - \frac{n-1}{u}\right) = 0 \qquad (31)$$

As a function of n only

$$\Delta \phi_n^u = \phi_{n+1}^u - \phi_n^u = \phi_n^u \left(\frac{u}{n+1} - 1\right) \qquad (32)$$

*See footnote on page 290.

and

$$\nabla\phi_n^u = \phi_n^u - \phi_{n-1}^u = \phi_n^u\left(1 - \frac{n}{u}\right) \quad (33)$$

Similarly

$$\Delta P_n^u = P_{n+1}^u - P_n^u = -\phi_n^u$$

$$\Delta^2 P_n^u = P_{n+2}^u - 2P_{n+1}^u + P_n^u$$

$$= \phi_n^u - \phi_{n+1}^u = \phi_n^u\left(1 - \frac{u}{n+1}\right)$$

$$\therefore\ \Delta^2 P_n^u + \Delta P_n^u\left(1 - \frac{u}{n+1}\right) = 0 \quad (34)$$

and

$$\nabla P_n^u = P_n^u - P_{n-1}^u = -\phi_{n-1}^u$$

$$\nabla^2 P_n^u = P_n^u - 2P_{n-1}^u + P_{n-2}^u$$

$$= -\phi_{n-1}^u + \phi_{n-2}^u = \phi_{n-1}^u\left(\frac{n-1}{n} - 1\right)$$

$$\therefore\ \nabla^2 P_n^u + \nabla P_n^u\left(\frac{n-1}{u} - 1\right) = 0 \quad (35)$$

From these relations one deduces that $-\phi_n^u$ is a particular solution for the following total-differential, total-difference, and partial-differential-difference equations

1. $$\frac{dy}{du} = y\left(\frac{n}{u} - 1\right) \quad (36)$$

2. $$\Delta y = y\left(\frac{u}{n+1} - 1\right) \quad (37)$$

3. $$\nabla y = y\left(1 - \frac{n}{u}\right) \quad (38)$$

4. $$\frac{\partial^r y}{\partial u^r} = (-1)^r \nabla^r y \quad (39)$$

In Equation (36) y is a continuous function of u only. In Equations (37) and (38) it is a step function of n only, and in Equation (39) it is a function of both u and n. One also deduces that $-P_n^u$ is a particular solution to the following equations:

1. $$\frac{d^2y}{du^2} + \frac{dy}{du}\left(1 - \frac{n-1}{u}\right) = 0 \quad (40)$$

2. $$\Delta^2 y + \Delta y\left(1 - \frac{u}{n+1}\right) = 0 \quad (41)$$

3. $$\nabla^2 y + \nabla y\left(\frac{n-1}{u} - 1\right) = 0 \quad (42)$$

4. $$\frac{\partial^r y}{\partial u^{-2}} = (-1)^r \nabla^r y \quad (39)^*$$

More relations can also be deduced by combining two or more of these relations.

*Since it is understood that differentiation is always concerned with the continuous variable u only, Equation (39) can also be written in the form of a total-differential-difference equation and can be called simply a *differential-difference equation.*

From (36) and (37) and similarly from (37) and (38), for example, one deduces that ϕ_n^u is also a solution to the following two equations:

$$\frac{\partial y}{\partial u} + \left[\frac{\frac{n}{u} - 1}{1 - \frac{u}{n+1}}\right] \Delta y = 0 \quad (43)$$

and

$$\Delta y + \frac{\left(\frac{u}{n+1} - 1\right)}{\frac{n}{u} - 1} \nabla y = 0 \quad (44)$$

AN ALTERNATIVE DERIVATION OF THE GENERAL ELUTION EQUATION

In a previous paper (*1*) the general elution equation was derived by first obtaining the expression for plate 1 and then going to plate 2 to derive its elution equation, and then to plate 3, and so on until the form of the solution became evident, when the general equation was deduced. An alternative derivation would be to set up the partial-differential-difference equation for plate n, write the general solution, and, using the boundary conditions of the problem, obtain the values of the constants.

If it is assumed that at the beginning of the elution process the concentrations on plates 1, 2, 3, . . . n, . . . N are

$$y_1^0,\ y_2^0,\ y_3^0 \cdots y_n^0, \cdots y_N^0,$$

a differential material balance around plate n gives

$$(ky_n - ky_{n-1})\, dw = \frac{-S}{N}\, dy_n$$

$$\therefore\quad k\nabla y_n\, dw = -\frac{S}{N}\, dy_n$$

Substitution of $x = wN/S$, $u = kx$ gives

$$\frac{dy_n}{du} = -\nabla y_n \quad (45)$$

According to Equation (39) both ϕ_n^u and P_n^u are particular solutions of this equation. A more general solution is

$$y_n = \sum_{n-m\geq 0} A_m \phi_{n-m}^u \quad (46)$$

m is an integer positive or negative such that $n - m \geq 0$. In order to evaluate the constants m and A_m, one substitutes the following boundary conditions:

1. $y_r = y_r^0$ when $u = 0$. Substituting in Equation (46) and making use of the fact that ϕ_{r-m}^0 is equal to 1 when $m = r$, is equal to 0 for $m < r$, and is undefined for $m > r$, one finds that $m = r$, $A_m = y_r^0$, and Equation (46) becomes

$$y_n = \sum_{n-r\geq 0} y_r^0 \phi_{n-r}^u \quad (47)$$

2. The second boundary condition is deduced from the fact that the concentration of solute in the eluent entering the first plate is equal to zero, and hence y_1 as a function of u can be deduced by a differential material balance around plate 1 resulting in

$$y_1 = y_1^0 e^{-u}$$

Substituting this boundary condition in Equation (47), one deduces that the solution is true only for values of r greater than or equal to 1, and the complete solution becomes

$$y_n = \sum_{\substack{n-r\geq 0 \\ r\geq 1}} y_r^0 \phi_{n-r}^u$$

or

$$y_n = \sum_{r=1}^{n} y_r^0 \phi_{n-r}^u \quad (48)$$

which is the general elution equation. This method can similarly be used to derive the elution equation for the special cases treated in the previous paper (*1*) and for deriving the deposition equation.

NOTATION

- k = adsorption or exchange coefficient when it is neither a function of x nor a function of y_n; $k = \bar{y}_n/y_n$
- N = total number of plates in column
- n = plate number from top of column. The top plate number $n = 1$ and the bottom plate number $n = N$.
- P_n^u = $\sum_{r=n}^{\infty} \phi_r^u$ = Poisson exponential summation
- R_n = y_n/y_0 or $\bar{y}_n/\bar{y}_0$
- R_n^0 = y_n^0/y_0
- S = total weight of adsorbent in the column
- u = kx during the elution process
- w = weight of eluent or solvent that has passed through any plate in the column
- x = wN/S
- y_n = concentration of solute on plate n, g. solute/g. adsorbent, for the elution process
- $\bar{y}_n$ = concentration of solute in eluent in equilibrium with plate n during elution
- $\bar{y}_0$ = concentration of solute in solvent before entering plate 1
- y_0 = concentration of solute on adsorbent if in equilibrium with solvent containing the solute at concentration $\bar{y}_0$
- ϕ_n^u = $e^{-u}u^n/n!$ = Poisson exponential function
- Δ = first forward finite difference
- Δ^r = rth forward finite difference
- ∇ = first backward finite difference
- ∇^r = rth backward finite difference

LITERATURE CITED

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